SOLUTION OF THE GENERAL CASE OF PROBLEM. 87, *MATH. VISITOR, NO.* 2.

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LET there be a series of casks containing a_1, a_2 etc. gallons, of which the first is filled with a mixture containing a gallons of wine, and the rest are filled with water. Water runs into the first cask at the rate of m gallons per minute, the mixture escapes into the second cask at the same rate, and the mixture in the second cask escapes into the third at the same rate, and so on. It is required to find the quantity of wine in each cask at the end of t minutes, the fluids being supposed to mingle perfectly.

Denoting the required quantities by x_1, x_2 etc., my published solution of the special case (Math. Visitor, Jan., 1879) suggests what will be verified in the following solution; namely, that the quantities take the form,

$$\left.\begin{array}{l}
x_1 = A_{1,1}\, e^{-\frac{mt}{a_1}} \\
x_2 = A_{1,2}\, e^{-\frac{mt}{a_1}} + A_{2,2}\, e^{-\frac{mt}{a_2}} \\
x_3 = A_{1,3}\, e^{-\frac{mt}{a_1}} + A_{2,3}\, e^{-\frac{mt}{a_2}} + A_{3,3}\, e^{-\frac{mt}{a_3}} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
x_n = A_{1,n}\, e^{-\frac{mt}{a_1}} + A_{2,n}\, e^{-\frac{mt}{a_2}} + \ldots\ldots + A_{n,n}\, e^{-\frac{mt}{a_n}}
\end{array}\right\} \qquad (1)$$

in which $A_{1,1}$ etc., are constants to be determined. The first subscript refers to the exponential to which the coefficient belongs and the second to the variable in whose expression it occurs. It is to be remarked that $A_{r,n}$ vanishes when $r > n$.

Since $x_1 \div a_1$ represents the fractional part of the mixture in the first cask which consists of wine, $m(x_1 \div a_1)$, $m(x_2 \div a_2)$, etc., denote the rates at which wine is escaping from the 1st, 2nd, etc. casks, and, at the same time, the rate at which wine is entering the 2nd, 3rd, etc. casks. Hence the conditions of the problem give the differential equations,

$$\left.\begin{array}{l}
\dfrac{dx_1}{dt} = \qquad\quad - \dfrac{mx_1}{a_1} \\
\dfrac{dx_2}{dt} = \dfrac{mx_1}{a_1} - \dfrac{mx_2}{a_2} \\
\dfrac{dx_3}{dt} = \dfrac{mx_2}{a_2} - \dfrac{mx_3}{a_3} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\dfrac{dx_n}{dt} = \dfrac{mx_{n-1}}{a_{n-1}} - \dfrac{mx_n}{a_n}
\end{array}\right\}, \qquad (2)$$

and, putting $t=0$ in (1), we have, to determine the constants of integration,

$$\left.\begin{array}{r} A_{1,1}=a \\ A_{1,2}+A_{2,2}=0 \\ A_{1,3}+A_{2,3}+A_{3,3}=0 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ A_{1,n}+A_{2,n}+A_{3,n}+\ldots+A_{n,n}=0 \end{array}\right\}. \qquad (3)$$

Differentiating the assumed values in (1), we have

$$\left.\begin{array}{l} \dfrac{dx_1}{dt}=-\dfrac{m}{a_1}A_{1,1}\,e^{-\frac{mt}{a_1}} \\ \dfrac{dx_2}{dt}=-\dfrac{m}{a_1}A_{1,2}\,e^{-\frac{mt}{a_1}}-\dfrac{m}{a_2}A_{2,2}\,e^{-\frac{mt}{a_2}} \\ \dfrac{dx_3}{dt}=-\dfrac{m}{a_1}A_{1,3}\,e^{-\frac{mt}{a_1}}-\dfrac{m}{a_2}A_{2,3}\,e^{-\frac{mt}{a_2}}-\dfrac{m}{a_3}A_{3,3}\,e^{-\frac{mt}{a_3}} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \dfrac{dx_n}{dt}=-\dfrac{m}{a_1}A_{1,n}\,e^{-\frac{mt}{a_1}}-\dfrac{m}{a_2}A_{2,n}\,e^{-\frac{mt}{a_2}}-\ldots-\dfrac{m}{a_n}A_{n,n}\,e^{-\frac{mt}{a_n}} \end{array}\right\}. \qquad (4)$$

If now we substitute the assumed values (1) in equations (2), and compare the results with equations (4), the first of these is verified; and, by equating the coefficients of the exponentials in the nth equations, we have

$$\left.\begin{array}{rcl} -\dfrac{m}{a_1}A_{1,n} &=& \dfrac{m}{a_{n-1}}A_{1,n-1}-\dfrac{m}{a_n}A_{1,n} \\ -\dfrac{m}{a_2}A_{2,n} &=& \dfrac{m}{a_{n-1}}A_{2,n-1}-\dfrac{m}{a_n}A_{2,n} \\ \cdot\;\cdot\;\cdot && \cdot\;\cdot\;\cdot\;\cdot \\ -\dfrac{m}{a_r}A_{r,n} &=& \dfrac{m}{a_{n-1}}A_{r,n-1}-\dfrac{m}{a_n}A_{r,n} \\ \cdot\;\cdot\;\cdot && \cdot\;\cdot\;\cdot\;\cdot \\ -\dfrac{m}{a_n}A_{n,n} &=& \qquad\qquad -\dfrac{m}{a_n}A_{n,n} \end{array}\right\} \qquad (5)$$

the last of which is an identity. The general equation in (5) reduces to

$$A_{r,n}=\frac{a_r\,a_n}{a_{n-1}\,(a_r-a_n)}A_{r,n-1},$$

by substituting the corresponding expression for $A_{r,n-1}$, we obtain

$$A_{r,n}=\frac{a_r^2\,a_n}{a_{n-2}\,(a_r-a_{n-1})(a_r-a_n)}A_{r,n-2},$$

etc. etc.; the $(n-r)$th equation being

$$A_{r,n}=\frac{a_r^{n-r-1}\,a^n A_{r,r}}{(a_r-a_{r+1})(a_r-a_{r+2})\ldots(a_r-a_n)}. \qquad (6)$$

By means of this relation and the group (3) the values of the constants

may be determined. Thus from (3) we have $A_{1,1} = a$, hence from (6)

$$A_{1,n} = \frac{a_1^{n-2}\, a_n a}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)}. \tag{7}$$

Again, making $n = 2$ in (7),

$$A_{1,2} = \frac{a_2 a}{a_1-a_2},$$

substituting in the second of group (3), we derive

$$A_{2,2} = \frac{a_2 a}{a_2-a_1},$$

whence, from (6),

$$A_{2,n} = \frac{a_2^{n-2}\, a_n a}{(a_2-a_1)(a_2-a_3)\ldots(a_2-a_n)}. \tag{8}$$

The form of equations (7) and (8) suggests that in general

$$A_{r,n} = \frac{a_r^{n-2}\, a_n a}{(a_r-a_1)(a_r-a_2)\ldots(a_r-a_n)} \tag{9}$$

(the subscripts of the second letter in the binomials taking all values from 1 to n except r), and this expression will be demonstrated when it is shown that it satisfies the nth equation of group (3); that is, the equation

$$A_{1,n} + A_{2,n} + A_{3,n} + \ldots. + A_{n,n} = 0.$$

Making the substitutions, dividing by $a_n a$, and multiplying by the expression

$$\begin{aligned}(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)\times(a_2-a_3)(a_2-a_4)\ldots(a_2-a_n)&\\ \times(a_3-a_4)\ldots(a_{n-1}-a_n)&,\end{aligned}$$

the equation to be verified becomes

$$\left.\begin{array}{l} +a_1^{n-2}(a_2-a_3)\ldots(a_2-a_n)\,.\,(a_3-a_4)\ldots(a_3-a_n)\ldots(a_{n-1}-a_n) \\ -a_2^{n-2}(a_1-a_3)(a_1-a_4)\ldots(a_1-a_n)\,.\,(a_3-a_4)\ldots(a_{n-1}-a_n) \\ +a_3^{n-2}(a_1-a_2)(a_1-a_4)\ldots(a_1-a_n)\ldots. \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \end{array}\right\} = 0,$$

in which the coefficient of a_r^{n-1} is the product of all the differences which do not involve a_r. Using the determinant expressions for these products, the equation takes the form

$$a_1^{n-2}\begin{vmatrix} 1, & 1 \ldots. & 1 \\ a_n, & a_{n-1}, \ldots & a_2 \\ a_n^2, & a_{n-1}^2, \ldots & a_2^2 \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_n^{n-2}, & \ldots\ldots & a_2^{n-2} \end{vmatrix} - a_2^{n-2}\begin{vmatrix} 1, & 1, \ldots & 1, & 1 \\ a_n, & \ldots & a_3, & a_1 \\ a_n^2, & \ldots & a_3^2, & a_1^2 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ a_n^{n-2}, & \ldots. & & a_1^{n-2} \end{vmatrix} + a_3^{n-2}\begin{vmatrix} 1; & 1 \ldots \\ a_n & \ldots \\ a_n^2 & \ldots \\ \vdots & \\ \vdots & \\ a_n^{n-2} & \ldots \end{vmatrix} \text{etc.} = 0.$$

Now the right hand member of this equation is equivalent to the determinant

$$\left|\begin{array}{llll} 1\;, & 1\;, & \ldots 1\;, & 1 \\ a_n\;, & a_{n-1}\;, & \ldots a_2\;, & a_1 \\ a_n^2\;, & a_{n-1}^2\;, & \ldots a_2^2\;, & a_1^2 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_n^{n-2}\;, & a_{n-1}^{n-2}\;, & \ldots a_2^{n-2}\;, & a_1^{n-2} \\ a_n^{n-2}\;, & a_{n-1}^{n-2}\;, & \ldots a_2^{n-2}\;, & a_1^{n-2} \end{array}\right|$$

which is identically $= 0$, because the last two rows are identical. Hence the solution of the problem is given by (1), the coefficients being determined by (9).

It is a remarkable result of this solution that the value of the fraction $x_n \div a_n$, or ratio of the wine to the whole contents of the last cask, is wholly independent of the order of the casks.

NOTE ON A FORM OF THE EQ. OF THE TANG. TO A CONIC.

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THE envelope of any line $A\alpha^2 + B\alpha + C = 0$, where α is an arbitrary parameter, and A, B and C are functions of x and y, is $B^2 = 4AC$; it follows that if the equation to any conic be put in the form $B^2 = 4AC$, the equation to the tangent may be written either as $A\alpha^2 + B\alpha + C = 0$ or $C\alpha^2 + B\alpha + A = 0$.

The equations to the conic sections and their tangents may be written as follows:

Circle,	$y^2 = (a+x)(a-x)$,	tangent,	$(a-x)\alpha^2 + 2y\alpha + a + x = 0$.
Parabola,	$y^2 = 4ax$,	"	$a\alpha^2 + y\alpha + x = 0$.
Ellipse,	$a^2y^2 = b^2(a+x)(a-x)$,	"	$b(a-x)\alpha^2 + 2ay\alpha + b(a+x) = 0$.
Hyperb.,	$a^2b^2 = (bx+ay)(bx-ay)$,	"	$(bx-ay)\alpha^2 + 2ab\alpha + bx + ay = 0$.

This form of the equation to the tangent will be found useful in certain problems of Analytical Geometry; thus, suppose it required to find the tangents to the conic

$$y^2 - 2xy + 2x^2 + 8y + 6x + 49 = 0,$$

which pass through $(-14, -22)$. The equation may be put in the form $(y - x + 4)^2 + (x + 11)(x + 3) = 0$, and the equation to the tangent is $(x+11)\alpha^2 + 2(y-x+4)\alpha - x - 3 = 0$; substituting the co-ordinates of the given point, we have $3\alpha^2 + 8\alpha - 11 = 0$, whence $\alpha = 1$ or $-\frac{11}{3}$, and the equations to the required tangents are $y - x + 8 = 0$, and $89x - 33y + 520 = 0$.